SMOOTH MANIFOLDS FALL 2022 - HOMEWORK 7

Problem 1. Let G be a connected Lie group, and $\mathfrak{h}$ be an ideal of $\mathrm{Lie}(G)$.

(a) Show that there is a well defined action $\overline{\mathrm{Ad}} : G \to GL(\mathrm{Lie}(G)/\mathfrak{h})$ defined by

$$\overline{\mathrm{Ad}}(g)(X + \mathfrak{h}) = \mathrm{Ad}(g)X + \mathfrak{h}.$$

(b) Show that the kernel of $\overline{\mathrm{Ad}}$ is the set $\{g \in G : \mathrm{Ad}(g)X \in \mathfrak{h} \text{ for all } X \in \mathrm{Lie}(G)\}$.

(c) Give an example of a Lie group and proper ideal for which Ad is faithful but $\overline{\mathrm{Ad}}$ is trivial.

Problem 2. If V is a vector space of dimension n and $\omega \in \Lambda^k(V)$ is an alternating k-multilinear function and $v \in V$, let $\iota_v\omega \in \Lambda^{k-1}(V)$ be the functional $(w_1, \ldots, w_{k-1}) \mapsto \omega(v, w_1, \ldots, w_{k-1})$. Let $\ker\omega = \{v \in V : \iota_v\omega = 0\}$.

(a) Show that $\ker\omega$ is a subspace of V.

(b) Prove or find a counterexample: for a fixed v, $\iota_v : \Lambda^2(V) \to \Lambda^1(V)$ is never onto.

(c) Prove or find a counterexample: if $W \subset \ker\omega$, then ω induces a well defined alternating k-multilinear function on V/W.

Problem 3. Let V be an n-dimensional vector space and $\omega \in \Lambda^k(V)$. If $W \subset V$ is a vector subspace, let $\pi_W : \Lambda^k(V) \to \Lambda^k(W)$ denote the restriction map, $\pi_W(\alpha) = \alpha|_W$.

(a) Show that π_W is linear and onto. Use this to compute $\dim(\ker\pi_W)$.

(b) Find a basis for $\ker\pi_W$ when $V = \mathbb{R}^5$, $W = \mathbb{R}^3 \times \{0\}$ and $k = 2$.